2025 WAMC 12 Solutions

1. What is the sum of the digits of

$$\frac{202503 + 202505 + 202512 + 202513 + 202517}{5}?$$

(A) 7 **(B)** 9 **(C)** 10 **(D)** 14 **(E)** 19

Ryan Tang

Solution: This is equal to

$$\frac{5 \cdot 202500 + 3 + 5 + 12 + 13 + 17}{5} = 202500 + 10 = 202510,$$

which has digit sum $\boxed{\textbf{(C)}\ 10}$.

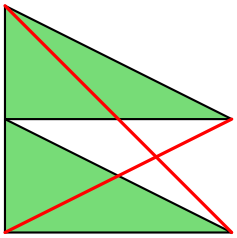
2. Points A, B, and C lie on circle ω with center O. If $\angle OAB = \angle OAC = 20°$, find $\angle OBC$.

(A) $10°$ **(B)** $20°$ **(C)** $40°$ **(D)** $50°$ **(E)** $70°$

Ryan Tang

Solution: We know that $\angle BAC = 20° + 20° = 40°$, so $\angle BOC = 2\angle BAC = 80°$. Because $OB = OC$, triangle BOC is isosceles, so $\angle OBC = \angle OCB = \frac{180° - 80°}{2} = \boxed{\textbf{(D)}\ 50°}$.

3. In the figure below, the sum of the squares of the lengths of the red segments is 10. The two congruent light blue right triangles have leg lengths in the ratio $1 : 2$. Find their total area.



(A) $\frac{6}{7}$ **(B)** $\frac{15}{13}$ **(C)** $\frac{9}{7}$ **(D)** $\frac{10}{7}$ **(E)** $\frac{20}{13}$

Ryan Tang

Solution: Let the leg lengths of the green triangles be a and $2a$. Then, the shorter line segment has length $\sqrt{a^2 + (2a)^2} = \sqrt{5a^2}$, and the longer line segment has length $\sqrt{(2a)^2 + (2a)^2} = \sqrt{8a^2}$. The sum of their squares is $13a^2 = 10$, so $a^2 = \frac{10}{13}$.

The area of the two triangles is $2a^2 = \boxed{\textbf{(E)}\ \frac{20}{13}}$.

4. Arjun, Vincent, Jacob, and Alexander stand in a line. Arjun refuses to stand in the first or second position. Vincent refuses to stand in the second or third position. Jacob refuses to stand in the third or fourth position.

Alexander refuses to stand in the fourth or first position. How many possible orderings are there?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Kaiyuan Mao

Solution: We casework based on where Arjun stands.
If he stands in the third position, then Alexander has to stand in the second position, so Jacob has to stand in the first position, and Vincent has to stand in the fourth position. This is one solution.
If Arjun stands in the fourth position, then Vincent has to stand in the first position, so Jacob has to stand in the second position, and Alexander has to stand in the third position. This is another solution.
We've exhausted all possible positions for Arjun, so the answer is $\boxed{\textbf{(C)}\ 2}$.

5. Ryan writes down the number 11111122. Patrick randomly rearranges the digits into a new eight-digit number. What is the probability that the sum of the digits in the left half equals the sum of the digits in the right half?

(A) $\frac{1}{4}$ **(B)** $\frac{3}{8}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{4}{7}$ **(E)** $\frac{5}{7}$

Ryan Tang

Solution: Note that for the two sides to have equal sums, both halves must contain 2 exactly once each. Let us place a 2 randomly. Without loss of generality, assume it lands in the first half. Then, there are 4 spots to place the next 2 in the other half, and there are 7 spots remaining. Therefore, the answer is $\boxed{\textbf{(D)}\ \frac{4}{7}}$.

6. Bruce rolls a die numbered 1 through 10. Each face of the die is weighted so that the probability of rolling n is directly proportional to n. What is the expected value of one spin?

(A) $\frac{11}{2}$ **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

Patrick Sun

Solution: The expected value is

$$1 \cdot \frac{1}{55} + 2 \cdot \frac{2}{55} + \ldots + 10 \cdot \frac{10}{55} = \frac{1^2 + 2^2 + \ldots + 10^2}{55} = \frac{10 \cdot 11 \cdot 21}{6 \cdot 55} = \boxed{\textbf{(C)}\ 7}.$$

7. In a math competition with 100 participants ranked from 1 to 100, Ryan randomly chooses an integer $N > 4$, then randomly interviews 2 of the top N participants. What is the probability that both the first and second-ranked participants are interviewed?

(A) $\frac{1}{2500}$ **(B)** $\frac{1}{1000}$ **(C)** $\frac{1}{500}$ **(D)** $\frac{1}{200}$ **(E)** $\frac{1}{100}$

Ryan Tang

Solution: Given a fixed N, the probability the first and second-ranked participants are interviewed is $\frac{1}{\binom{N}{2}} = \frac{2}{N(N-1)} = \frac{2}{N-1} - \frac{2}{N}$. Thus, as N ranges from 5 to 100, our sum is $\frac{2}{4} - \frac{2}{5} + \frac{2}{5} - \frac{2}{6} + \ldots + \frac{2}{99} - \frac{2}{100} = \frac{12}{25}$. However, we must divide by 96, as each N has a $\frac{1}{96}$ probability of being selected, and this gives us our final answer of $\boxed{\textbf{(D)}\ \frac{1}{200}}$.

8. Let p be a prime for which there exist positive integers a and b satisfying $\sqrt{a^2+p} = b^2$. Find the sum of the four smallest possible values of p.

(A) 104 **(B)** 116 **(C)** 126 **(D)** 138 **(E)** 144

Ryan Tang

Solution: Squaring both sides and rearranging yields $p = b^4 - a^2 = (b^2-a)(b^2+a)$. Since p is a prime, $b^2 - a$ must equal 1, and hence $a = b^2 - 1$ and $p = b^2 + a = 2b^2 - 1$. Then, simply testing values of b and checking for primality allows us to determine that $b = 2, 3, 4, 6$ result in $p = 7, 17, 31, 71$, which sum to $\boxed{\textbf{(D)}\ 126}$.

9. For how many integers $1 \leq n \leq 10^{100}$ is

$$\log_2 \log_5 n - \log_2 \log_5 10$$

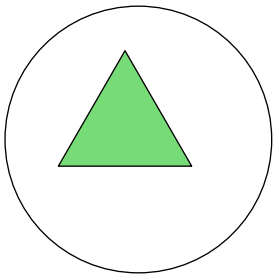
an integer?

(A) 0 **(B)** 1 **(C)** 3 **(D)** 6 **(E)** 7

Ryan Tang & Wesley Wu

Solution: Using logarithm rules, we can simplify the expression into $\log_2 \frac{\log_5 n}{\log_5 10} = \log_2 \log_{10} n$. Then, for this to be an integer, n must equal 10^{2^k}, where k is also an integer. Then, $2^k \leq 100$, so k ranges from 0 to 6, inclusive. There are thus $\boxed{\textbf{(E)}\ 7}$ valid numbers.

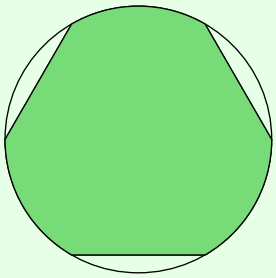
10. Laura takes a sponge in the shape of an equilateral triangle with side length 2 and dips it in paint. Then, without rotating the sponge or painting outside the circle, she paints as much of the interior of a circle with radius 2 as she can. How much of the circle is left unpainted?



(A) $2\pi - 3\sqrt{3}$ **(B)** $2\pi - 3\sqrt{2}$ **(C)** $2\pi - 2\sqrt{3}$ **(D)** $4\pi - 3\sqrt{3}$ **(E)** $4\pi - 2\sqrt{3}$

Laura Wang

Solution: Through observation, note that the following shaded region is the region of every part of the circle that can be painted.



Each of the small unshaded areas is created by the subtraction of an equilateral triangle from a sector. Each sector has central angle $60°$, so each one has area $\frac{2}{3}\pi$; each equilateral triangle has side length 2 and area $\sqrt{3}$. Thus, our answer is $3\left(\frac{2}{3}\pi - \sqrt{3}\right) = \boxed{\textbf{(A) } 2\pi - 3\sqrt{3}}$.
Note: We really should have put $4\pi - \sqrt{3}$ as a solution since it would have baited the many people who failed to read the question.

11. Ryan places a standard fair 6-sided die (with opposite faces summing to 7) on the table with the 1 face on top. He then rolls the die to an adjacent face three times in succession. What is the probability that no number greater than 3 ever appears on the top?

(A) $\frac{1}{216}$ **(B)** $\frac{1}{64}$ **(C)** $\frac{1}{9}$ **(D)** $\frac{1}{8}$ **(E)** $\frac{1}{6}$

Ryan Tang

Solution: Each turn, there are four options to roll the die to. Note that the three numbers that satisfy the condition, 1, 2, and 3, can all be reached from the other two. As we start on 1, for each step, there are 2 valid moves of the total 4. The probability of this happening three times in a row is $\left(\frac{1}{2}\right)^3 = \boxed{\textbf{(D) } \frac{1}{8}}$.

12. What is the area formed by all coordinate pairs (x, y) that satisfy

$$(x - x_1)^2 + (y - y_1)^2 \leq 4$$

over all x_1 and y_1 that satisfy $x_1^2 + y_1^2 = 200$?

(A) $80\pi\sqrt{2}$ **(B)** $100\pi\sqrt{2}$ **(C)** 200π **(D)** 204π **(E)** $160\pi\sqrt{2}$

Alexander Yang

Solution: For each (x_1, y_1) on the circle $x_1^2 + y_1^2 = 200$, the condition shades the disk of radius 2 centered at (x_1, y_1). The union of those disks is the annulus of all points whose distance r from the origin satisfies

$$\sqrt{200} - 2 \leq r \leq \sqrt{200} + 2.$$

The area is thus

$$\pi\left((\sqrt{200}+2)^2 - (\sqrt{200}-2)^2\right) = \pi \cdot 8\sqrt{200} = 8\pi\sqrt{200} = \boxed{\textbf{(A) } 80\sqrt{2}\pi}.$$

13. Triangle $\triangle ABC$ intersects circle ω at points D, E, F, G, H, I such that $ADEB$, $BFGC$, and $CHIA$ are collinear in those orders. Given that $AD = 3$, $AI = 4$, $HC = 4$, $GC = 5$, and $DE + FG = 8$, find the value of DE.

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

Ryan Tang

Solution: By Power of a Point, $AD \cdot AE = AI \cdot AH$, and since $AI = CH$ and $AH = CI$, we can manipulate this into $AD \cdot AE = CH \cdot CI = CG \cdot CF$ by Power of a Point again. Plugging our values in, this becomes $3AE = 5CF$.
Now, we set $DE = x$ and $FG = 8 - x$. Our equation becomes $3(3 + x) = 5(5 + 8 - x)$, and solving this yields $\boxed{\textbf{(E)}\ 7}$.

14. How many ways are there to rearrange the letters in the word NAGGING so that no G's are adjacent?

(A) 48 **(B)** 60 **(C)** 120 **(D)** 168 **(E)** 240

Wesley Wu

Solution: Consider the possible positions of the three G's. There are ten possible locations of the G, as detailed below. They can also be counted systematically through $(3 + 2 + 1) + (2 + 1) + (1)$, achieved using casework on the positions of the first and second G.

G_G_G__	G___G_G
G_G__G_	_G_G_G_
G_G___G	_G_G__G
G__G_G_	_G__G_G
G__G__G	__G_G_G

The other four letters can be placed into the blanks in $\frac{4!}{2!} = 12$ ways (to account for the duplicate N), so the answer is $12 \cdot 10 = \boxed{\textbf{(C)}\ 120}$.

Alternate Solution: We proceed with complementary counting. To count the number of cases where two G's are together, we treat them as a combined letter. We compute the number of ways to arrange $NA[GG]ING$, which is $\frac{6!}{2!} = 360$. However, this overcounts the case where there are 3 G's in a row. This is counted by the number of ways to arrange $NA[GGG]IN$, which is $\frac{5!}{2!} = 60$. Thus, the number of cases with adjacent G's is $360 - 60 = 300$, and the answer is $\frac{7!}{3!2!} - 300 = \boxed{\textbf{(C)}\ 120}$.

15. The solutions to the equation $x^6+x^5+x^4+2x^3+x^2+x+1 = 0$ can be expressed as $e^{2i\pi a_k}$, where $1 \leq k \leq 6$ and $0 < a_k < 1$. Compute $a_1 + a_2 + a_3 + a_4 + a_5 + a_6$.

(A) 0 **(B)** 2 **(C)** 3 **(D)** 6 **(E)** 12

Wesley Wu

Solution: We factor the left-hand side as $(x^3+x^2+x+1)(x^3+1)$, from which we extract that the roots of the original equation are the roots to $\frac{x^4-1}{x-1}$ and $\frac{x^6-1}{x^3-1}$. Thus, the solution set is the union of the fourth and sixth roots of unity, with a double root at $x=-1$ and excluding 1 and the 3rd roots of unity.
This means that our a_k are $\frac{1}{6}, \frac{3}{6}=\frac{1}{2}, \frac{5}{6}, \frac{1}{4}, \frac{2}{4}=\frac{1}{2}$, and $\frac{3}{4}$. Summing these up gives $\boxed{\textbf{(C) } 3}$.
Note: In the problems document, there was an errata in which we typed $0 < a_k < 2\pi$ instead. The typo has been corrected here.

16. Real numbers x and y satisfy the system of equations

$$\begin{aligned} x^3-8x^2+px-q&=0,\\ y^3-8y^2+py-q&=0, \text{ and}\\ (x+y)^3-8(x+y)^2+p(x+y)-q&=0. \end{aligned}$$

There exists a unique k for which $kp+q$ is constant for all values of x and y. Find the value of $kp+q$.

(A) -64 **(B)** -58 **(C)** -32 **(D)** -10 **(E)** 26

Ryan Tang

Solution: Note that x, y, and $x+y$ are all roots of n^3-8n^2+pn-q. Then, $x+y+(x+y)=8$ by Vieta's, so $x+y=4$. Then, plugging this into the last equation gives $-64=(-4)p+q$, so $k=-4$ and $kp+q=\boxed{\textbf{(A) } -64}$.

17. Chords AB and AC of a circle with center O have lengths 10 and 16 and midpoints M and N, respectively. Line MN is extended to meet the circle at X and Y, so that X and N are on opposite sides of AB. Let Z be the reflection of X over M. If $\angle MON = 120^\circ$, compute YZ.

(A) $\frac{44}{7}$ **(B)** $\frac{88}{7}$ **(C)** 14 **(D)** 20 **(E)** $\frac{160}{7}$

Wesley Wu & Ryan Tang

Solution: Since $MONA$ is cyclic, $\angle MAN = 60^\circ$, and since $AM=5$ and $AN=8$, $MN=7$ by Law of Cosines.
Let $MX=a$ and $NY=b$. Using power of a point on point M gives $5\cdot 5 = a(7+b) = ab+7a$, and using it on point N gives $8\cdot 8 = b(7+a) = ab+7b$. Note that the length of YZ is equal to $XY-2XM = a+b+7-2a = 7+b-a$. Then, subtracting the two PoP equations yields $7(b-a)=39$, so $YZ = 7+\frac{39}{7} = \boxed{\textbf{(B) } \frac{88}{7}}$.

18. Four consecutive natural numbers a, b, c, d satisfy $a<b<c<d$. If a is divisible by 3, b is divisible by 5, c is divisible by 7, and d is divisible by 9, find the minimum value of $a+b+c+d$.

(A) 82 **(B)** 106 **(C)** 510 **(D)** 642 **(E)** 654

Kaiyuan Mao

Solution: Let $a = n$. The four conditions can be rewritten as

$$n \equiv 0 \pmod 3, \ \ n+1 \equiv 0 \pmod 5, \ \ n+2 \equiv 0 \pmod 7, \text{ and } n+3 \equiv 0 \pmod 9.$$

The congruence $n+3 \equiv 0 \pmod 9$ has solution $n \equiv 6 \pmod 9$, which already implies $n \equiv 0 \pmod 3$, so we may instead work with the system

$$n \equiv -1 \equiv 4 \pmod 5, \ \ n \equiv -2 \equiv 5 \pmod 7, \text{ and } n \equiv 6 \pmod 9.$$

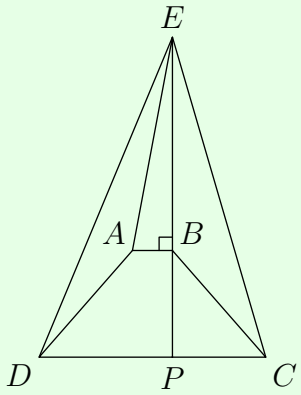
Now, let $n = 9k + 6$. The first condition gives us $9k + 6 \equiv 4 \pmod 5 \implies k \equiv 2 \pmod 5$, and the second yields $9k + 6 \equiv 5 \pmod 7 \implies k \equiv 3 \pmod 7$. The third is satisfied through our substitution. By combining the two, we obtain $k \equiv 17 \pmod{35}$, so the minimum k is 17, and the minimum n is 159. Extracting, our answer is $159 + 160 + 161 + 162 = \boxed{\textbf{(D)}\ 642}$.

19. Isosceles trapezoid $ABCD$ has $AB \parallel CD$ and $AB = 3$. Triangle ABE is drawn outside the trapezoid with $\angle ABE = 90^\circ$. If $EC = 25$, $ED = 26$, and the area of pentagon $AEBCD$ is 104, compute the area of triangle BDE.

(A) 54 **(B)** 76 **(C)** 80 **(D)** 104 **(E)** 204

Wesley Wu

Solution: Below is a diagram with the height from B to CD dropped.



Since $ABCD$ is isosceles, $DP = CP + AB = CP + 3$. Then, applying Pythagoras on DPE and CPE yields $26^2 - (CP+3)^2 = 25^2 - CP^2$, which we can solve for to get $CP = 7$, $DP = 10$, and $EP = 24$. The area of BDE can be expressed as $\frac{1}{2} \cdot EB \cdot DP = 5EB$. Let $n = EB$. Then, the area of EAB is $\frac{3n}{2}$, and the area of $ABCD$ is $10BP = 10(24 - n) = 240 - 10n$. Then, $240 - 10n + \frac{3n}{2} = 104$, which has solution $n = 16$. Therefore, our answer is $\boxed{\textbf{(C)}\ 80}$.

20. For how many integers n between 1 and 100, inclusive, is $\frac{n!^5}{5^n}$ an integer?

(A) 25 **(B)** 30 **(C)** 50 **(D)** 75 **(E)** 80

Ryan Tang

Solution: For $1 \le n < 25$, $n!$ has $\left\lfloor \frac{n}{5} \right\rfloor$ factors of 5, so $n!^5$ has $5\left\lfloor \frac{n}{5} \right\rfloor$ factors of 5, while 5^n has n factors of 5. For this quotient to be an integer, 5 must divide into n. This results in $n = 5, 10, 15, 20$, with 4 solutions.
For $n \ge 25$, $n!^5$ has $5(\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{25} \right\rfloor + \cdots)$ factors of 5, while 5^n has n factors of 5. Note that

$$5\left(\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{25} \right\rfloor + \cdots\right) \ge 5\left(\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{25} \right\rfloor\right) \ge 5\left(1 + \left\lfloor \frac{n}{5} \right\rfloor\right) \ge 5\left(\frac{n-4}{5} + 1\right) = n + 1 > n,$$

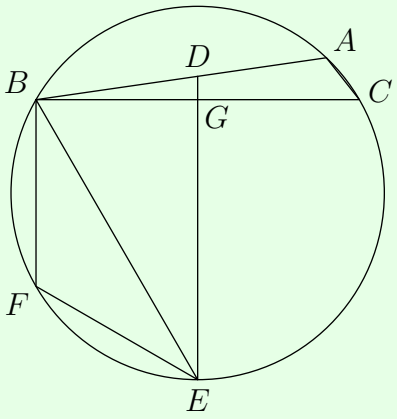
so the quotient is always an integer. This case has $100 - 25 + 1 = 76$ solutions.
Thus, the answer is $4 + 76 = \boxed{\textbf{(E)}\ 80}$.

21. Triangle $\triangle ABC$ has $AB = 11$, $AC = 2$, and circumcircle ω. Let D be a point on side AB, and point E is a point on arc BC of ω not including A satisfying $DE \perp BC$. Another point F on ω satisfies $\angle CBF = 90^\circ$. Given that $BE = BF\sqrt{3} = EF\sqrt{3}$, find BD.

(A) $\frac{41}{8}$ **(B)** $\frac{49}{8}$ **(C)** $\frac{49\sqrt{3}}{12}$ **(D)** $\frac{75}{8}$ **(E)** $7\sqrt{3}$

Ryan Tang

Solution: Let us draw a diagram for this complicated mess.



First and foremost, we note that BFE is a $30 - 30 - 120$ degree triangle. This means that $\angle BCE = 180^\circ - \angle BFE = 60^\circ$, and in addition, $\angle BEC = \angle CEF - \angle BEF = 90^\circ - 30^\circ = 60^\circ$, so BCE is equilateral.
Using cyclic quadrilateral $ABEC$, we must have $\angle BAC = 120^\circ$, so by Law of Cosines we find that $BC = 7\sqrt{3}$. Thus, since BG is a height of the equilateral, it must be half of BC, giving us $BG = \frac{7\sqrt{3}}{2}$. One more application of Law of Cosines yields $\cos \angle ABC = \frac{12}{7\sqrt{3}}$, and finally,

$$BD = \frac{BG}{\cos \angle DBG} = \frac{\frac{7\sqrt{3}}{2}}{\frac{12}{7\sqrt{3}}} = \boxed{\textbf{(B)}\ \frac{49}{8}}.$$

22. How many ordered pairs (a, b) are there to the equation

$$\gcd(a,b)^3 + 90\gcd(a,b) = (\gcd(a,b) + 1)(b - a)$$

if $1 \le a \le 500$?

(A) 42 **(B)** 43 **(C)** 70 **(D)** 98 **(E)** 129

Ryan Tang

Solution: Let $d = \gcd(a, b)$ so that $a = dx$ and $b = dy$ with $\gcd(x, y) = 1$. Since both sides are positive, we must have $y > x$, too. The equation becomes

$$d^3 + 90d = d(d+1)(y-x) \implies d^2 + 90 = (d+1)(y-x).$$

Hence,

$$y - x = \frac{d^2+90}{d+1} = d - 1 + \frac{91}{d+1},$$

so $d + 1 \mid 91$. Then, $d + 1$ must be among $\{1, 7, 13, 91\}$ and d must be positive, d must be $6, 12$, or 90. We can calculate that k is equal to $18, 18$, and 90, respectively.

For each d, we must count the number of positive integers x with $dx \leq 500$ and $\gcd(x, k) = 1$, which fixes y as well. Therefore, the number of solutions is the number of positive integers $x \leq \left\lfloor \frac{500}{d} \right\rfloor$ with $\gcd(x, k) = 1$.

For $d = 6$, of every $k = 18$ numbers, there must be $\phi(k) = 6$ x satisfying the condition. There are $\left\lfloor \frac{500}{6} \right\rfloor = 83$ unfiltered choices of x, and there are $\left\lfloor \frac{83}{18} \right\rfloor = 4$ periods of 18, resulting in $4 \cdot 6 = 24$ cases. The remaining numbers to consider are $72 < x \leq 83$, of which only $x = 73, 77, 79, 83$ are coprime to k. Adding these gives a total count of 28 for this case.

For $d = 12$, there must be 6 x satisfying the condition every 18 numbers still. There are $\left\lfloor \frac{500}{12} \right\rfloor = 41$ unfiltered choices of x, and there are $\left\lfloor \frac{41}{18} \right\rfloor = 2$ periods of 18, resulting in $2 \cdot 6 = 12$ cases. The remaining numbers are $36 < x \leq 41$, and only $x = 37, 41$ work here, for a total of 14 numbers for this case.

Lastly, for $d = 90$, there are only $\left\lfloor \frac{500}{90} \right\rfloor = 5$ original choices for x. The only x satisfying $\gcd(x, 90) = 1$ is $x = 1$, so there is only one case here.

Summing up these results in $28 + 14 + 1 = \boxed{\textbf{(B)}\ 43}$.

23. Five cubes with side lengths 1, 2, 3, 4, 5 are stacked vertically so that the centers of touching faces align. The surface area of a tower includes all exposed faces, including the bottom face. Let S be the sum of the surface areas of all $5! = 120$ possible towers. What is S?

(A) 31200 **(B)** 33600 **(C)** 34800 **(D)** 36000 **(E)** 38400

Kaiyuan Mao

Solution: Without considering overlaps, the surface area of all 120 permutations would be $120 \cdot 6(1^2 + 2^2 + 3^2 + 4^2 + 5^2) = 39600$.
Let us consider the expected area that is obscured. Note that this expected area multiplied by 120 will provide the total covered area over all permutations. Consider the box with side length 1. This box's upper face neighbors the other 4 boxes with probability $\frac{1}{5}$ each and is obscured every time (except when it is the topmost cube), so the total surface area lost is $\frac{2}{5} \cdot 4 \cdot 1^2$. Similarly, the box with side length 2 adds $\frac{2}{5} \cdot 3 \cdot 2^2$ to the lost area, and we repeat this process for the 3 to 5 boxes. However, we must also account for the surface areas lost to the bottom faces, which is symmetric, and thus our expected obscured area is

$$\frac{4}{5} \cdot (4 \cdot 1^2 + 3 \cdot 2^2 + 2 \cdot 3^2 + 1 \cdot 4^2) = 40.$$

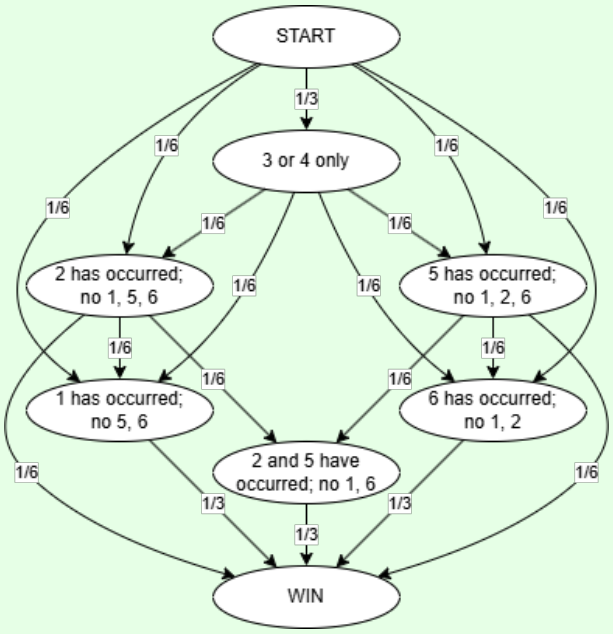
Multiplying this over all 120 permutations yields 4800, and subtracting this from 39600 results in the final answer of $\boxed{\textbf{(C)}\ 34800}$.

24. A fair six-sided die is rolled repeatedly. Let m_k and M_k be the minimum and maximum rolls, respectively, after k rolls of the die. Let n be the least possible n such that $M_n - m_n \geq 4$. Find the expected value of n.

(A) $\frac{7}{2}$ **(B)** 4 **(C)** 5 **(D)** $\frac{23}{4}$ **(E)** 6

Patrick Sun

Solution: Below is a diagram of all the states and their connections to each other; self-connections are omitted for clarity.



Note that the two sides are essentially symmetric. Furthermore, the branches from START and "3 or 4 only" are equivalent; hence, we assign the five rows (excluding START) values A, B, C, D, and E from top to bottom, and the value of START happens to be equal to A. These values count the expected number of moves to travel from those states to the WIN condition–this means that $E = 0$. Now, we consider each state separately.

We have the equations

$$\begin{cases} E = 0 \\ D = \frac{2}{3}D + \frac{1}{3}E + 1 \\ C = \frac{2}{3}C + \frac{1}{3}E + 1 \\ B = \frac{1}{2}B + \frac{1}{6}C + \frac{1}{6}D + \frac{1}{6}E + 1 \\ A = \frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C + 1 \\ X = A \end{cases}$$

Solving, we find that

$$D = 3, C = 3, B = 4, \text{ and } A = X = \boxed{\textbf{(C)}\ 5}.$$

25. A real number a is randomly selected from 0 to 1. What is the expected value of the number of real roots of the polynomial

$$x^9 + 3ax^6 + 3a^2x^3 - x + a^3 + a?$$

(A) $\frac{2\sqrt{3}}{3}$ **(B)** $\frac{4\sqrt{3}+9}{9}$ **(C)** $\frac{9\sqrt{3}-3}{5}$ **(D)** $\frac{5\sqrt{3}+5}{3}$ **(E)** 6

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Solution: We rewrite the polynomial as $(x^3+a)^3-(x-a)$. Its roots are the solutions to $(x^3+a)^3 = x-a$ or $x^3 + a = \sqrt[3]{x-a}$. These two are inverse functions (a.k.a. reflections over $y = x$), so it follows that the solutions to this equation only occur when $x^3 + a = x$.

Now, we need to find the number of roots to $x^3 - x + a = 0$. Using the shape of a cubic function, there are two values of a for which there is a double root. Between these two double roots, there will be three roots; outside of that range, there will be one. Thus, we set $x^3 - x + a = (x - r)^2(x - s)$, from which we get Vieta's relations $2r + s = 0$, $a = -r^2s$, and $r^2 + 2rs = -1$. Substituting $s = -2r$, the latter equation becomes $-3r^2 = -1$, so $r = \pm\sqrt{\frac{1}{3}}$ and $s = \mp 2\sqrt{\frac{1}{3}}$. Then, $a = \pm\frac{2}{3}\sqrt{\frac{1}{3}} = \pm\frac{2\sqrt{3}}{9}$.

We have the limit $0 < a < 1$, so when $0 < a < \frac{2\sqrt{3}}{9}$, there are three roots, and when $\frac{2\sqrt{3}}{9} < a < 1$, there is one root (we do not need to mention the case with two roots, as the probability of that happening is infinitesimally small). Thus, our expected value is

$$3 \cdot \frac{2\sqrt{3}}{9} + 1 \cdot \left(1 - \frac{2\sqrt{3}}{9}\right) = 1 + \frac{4\sqrt{3}}{9} = \boxed{\textbf{(B)}\ \frac{4\sqrt{3}+9}{9}}.$$

Note: It is much easier to find the double roots with calculus. Taking the derivative of $x^3 - x + a$ (which gives $3x^2 - 1$) and setting it to zero will give the x-values of the two tangency points with slope zero; plugging these two values back into x^3-x+a will allow us to tune a so that these two slope-zero tangency points land on the x-axis itself, creating a double root.